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Rational R-matrices in irreducible representations

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Abstract. Working directly from the Yang-Baxter equation, we obtain the spectral decomposition of rational R-matrices in irreducible representations, together with a necessary condition for their existence. Examples are given, and connections with Drinfeld's Yangian construction are discussed.

1. Introduction

This paper is concerned with the existence and form of rational solutions of the Yang-Baxter equation

$$R_{12}(u)R_{13}(u+v)R_{23}(v)=R_{23}(v)R_{13}(u+v)R_{12}(u) \qquad (u, v\in\mathbb{C}) \tag{1.1}$$

which takes values in $\mathrm{End}(V_1\otimes V_2\otimes V_3)$. The subscripts denote the spaces on which R acts:

$$R_{ab}(u): V_a\otimes V_b \to V_a\otimes V_b.$$

R is normally taken to act in representations $\rho_a\otimes\rho_b$ of a Lie algebra $\mathscr{A}$, so that V_a, V_b are the appropriate representation spaces of $\mathscr{A}$.

There are basically three types of solutions (R-matrices), classified according to their dependence on the spectral parameter u. The *elliptic* R-matrices are elliptic functions of u, and depend in addition on two further parameters. In an appropriate limit of one of these parameters, we obtain the *trigonometric* solutions, which are exponential functions of u, and also depend on the remaining additional parameter q. The trigonometric solutions are related to the non-commutative, non-commutative Hopf algebras that have come to be known as quantum groups [1], and which are q-dependent deformations of Lie algebras. The third class of R-matrices, with which we are concerned, have *rational* dependence on u. They can be obtained by taking the $q\to 1$ limit of the trigonometric solutions, but here we study them in their own right. For a review of solutions of the Yang-Baxter equation see Jimbo [2].

There have been previous investigations of rational R-matrices. Many were concerned with the construction of R-matrices for specific algebras and representations. The seminal paper for a more general approach was that of Kulish *et al* [3]. In their paper, the form of solutions for the A_n series was derived; they also introduced a general technique, which has become known as the fusion procedure, for the construction of new R-matrices from existing ones.

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The most general approach to the subject is that of Drinfield [4]. He has related the rational R-matrices to a non-commutative, non-cocommutative Hopf algebra he named the Yangian, and was able to use the representation theory of the Yangian to work out, for general $\mathscr{A}$, some representations for which R-matrices must exist. He also rederived the form for the A_n solutions deduced by Kulish *et al.*

This paper is arranged as follows. In section 2, we work directly from the Yang-Baxter equation to derive, for general $\mathscr{A}$, a set of equations (2.11) which all unitary R-matrices must satisfy. When a solution of this set of equations exists, it gives us the spectral decomposition of the R-matrix, which we believe has not been obtained in this form previously [5], although similar results have been found by Ogievetsky and Wiegmann [6]. When a solution does not exist, there can be no R-matrix, and so a necessary condition for the existence of an R-matrix is that there should be a solution.

Our method is similar to that of Kulish *et al*, and our results reduce to theirs for the A_n series, and match other R-matrices previously found for individual algebras and representations. As an illustrative example we calculate new R-matrices, which Drinfeld showed must exist, in symmetric, traceless representations of SO(N). Unfortunately, we are unable to show that our condition is also sufficient, in the sense that the existence of a solution of our equations is not a guarantee of the existence of a solution of (1.1). Neither are we able to give a general method for finding those representations for which it holds. For this reason, we go on to discuss Drinfeld's Yangian construction.

In section 3, we describe the Yangian, and how Drinfeld was able to discover a general characterization of some, but not necessarily all, of the representations for which R must exist. A case by case analysis of our equations shows that they are soluble for those representations specified by Drinfeld, thus guaranteeing the existence of the R-matrices whose spectral decomposition was computed in section 2. Conversely, we have never found a solution to exist for any other representation, and we therefore believe that Drinfeld's set of representations is exhaustive.

Interestingly, it turns out that our equations can also be derived from the Yangian, although this does not appear to have been done before [5]. We believe that, if a general method for solving them could be found, it would give the complete classification and spectral decomposition of unitary, rational R-matrices.

In conclusion, we mention briefly the fusion procedure [3], which also gives an indication of the irreducible representations for which it is possible to find rational R-matrices, and the more general question of solutions in reducible representations.

2. Rational R-matrices in irreducible representations

We seek rational solutions of the Yang–Baxter equation, (1.1), that are both unitary, $R(u)R(-u)=1$, and have $R(u)\to 1$ as $u\to\infty$. We can then write

$$R(u)=1+r(u)+\mathrm{O}\left(\frac{1}{u^2}\right) \tag{2.1}$$

and $r(u)$ must now satisfy the classical Yang–Baxter equation,

$$[r_{12}(u), r_{13}(u+v)]+[r_{12}(u), r_{23}(v)]+[r_{13}(u+v), r_{23}(v)]=0 \tag{2.2}$$

obtained by substituting (2.1) into (1.1) and examining the leading term, of order $1/u^2$.

We examine R-matrices which have as their classical limit

$$r(u)=\frac{1}{u}I_\mu\otimes I_\mu$$

where I_μ are the generators of $\mathscr{A}$. (Summation over repeated indices is always implied.) This is the simplest rational solution of (2.2). For a review of solutions of the classical Yang–Baxter equation, see Belavin and Drinfeld [7].

First, we write

$$R(u)=1+r(u)+\frac{X}{u^2}+\mathrm{O}\left(\frac{1}{u^3}\right).$$

Next, we use the unitarity condition to find X. Upon examining $R(u)R(-u)=1$ as $u\to\infty$, we find that

$$X=\tfrac{1}{2}I_\mu I_\nu\otimes I_\mu I_\nu. \tag{2.3}$$

Our strategy is now to examine the $v\to\infty$ limit of (1.1). Doing this, we obtain

$$\begin{aligned}R_{12}(u)&\left(1+\frac{1}{u+v}I_\mu\otimes 1\otimes I_\mu+\frac{1}{(u+v)^2}X_{13}\right)\left(1+\frac{1}{v}1\otimes I_\nu\otimes I_\nu+\frac{1}{v^2}X_{23}\right)\\&=\left(1+\frac{1}{v}1\otimes I_\nu\otimes I_\nu+\frac{1}{v^2}X_{23}\right)\\&\quad\times\left(1+\frac{1}{u+v}I_\mu\otimes 1\otimes I_\mu+\frac{1}{(u+v)^2}X_{13}\right)R_{12}(u)+\mathrm{O}\left(\frac{1}{v^3}\right).\end{aligned}\tag{2.4}$$

It should be noted that here we are implicitly assuming the existence of R_{13} and R_{23} for some V_3. This is trivial if $V_1=V_2=V_3=a$ (say), but if $V_1=V_3=a$ and $V_2=b$ then we are assuming that if R_{ab} exists, so does R_{aa}.

For R to be a solution of (1.1), it must satisfy (2.4) at each order of $1/v$, so we now expand out the brackets and equate coefficients of 1, $1/v$ and $1/v^2$. The results of this can most easily be een by multiplying through by $v(u+v)$. The left-hand side of (2.4) then becomes

$$\begin{aligned}R_{12}(u)\bigg(&v(u+v)+v(1\otimes I_\mu+I_\mu\otimes 1)\otimes I_\mu+u1\otimes I_\mu\otimes I_\mu\\&+I_\mu\otimes I_\nu\otimes I_\mu I_\nu+\frac{u+v}{v}X_{23}+\frac{v}{u+v}X_{13}\bigg).\end{aligned}$$

We now see that the terms of order v^2 are trivially equal, while those of order v give†

$$[R(u),1\otimes I_\mu+I_\mu\otimes 1]=0 \tag{2.5}$$

which expresses the invariance of R under the diagonal action of $\mathscr{A}$. Hence, by Schur's lemma, we can express $R_{ab}(u)$ in the form

$$R_{ab}(u)=\sum_{c\subset a\otimes b}\tau_c(u)P_c \tag{2.6}$$

† I am grateful to A J Macfarlane for pointing this out to me in the SU(2) case.

where P_c is the projector onto the irreducible component c of the tensor product $a\otimes b$, and τ_c are some rational functions of u, as yet undetermined. At this stage, we should point out that we are only able to apply our analysis to decompositions without multiplicities since, otherwise, R acts on the isomorphic components $c^1,\ldots,c^r$ as a matrix $M_{\alpha\beta}(u)$ (where $\alpha,\beta=1,\ldots,r$), and M will not, in general, be diagonalizable.

The terms of order 1 are given by

$$R_{12}(u)(u1\otimes I_\mu\otimes I_\mu+I_\mu\otimes I_\nu\otimes I_\mu I_\nu+X_{13}+X_{23})$$
$$=(u1\otimes I_\mu\otimes I_\mu+I_\mu\otimes I_\nu I_\mu+X_{13}+X_{23})R_{12}(u). \tag{2.7}$$

We can simplify this by noticing that

$$X_{13}+X_{23}=\tfrac{1}{2}\{(1\otimes I_\mu+I_\mu\otimes 1)(1\otimes I_\nu+I_\nu\otimes 1)-I_\mu\otimes I_\nu-I_\nu\otimes I_\mu\}\otimes I_\mu I_\nu$$

so that, because of (2.5), we can rewrite (2.7) as

$$R_{12}(u)(u1\otimes I_\mu I_\mu+\tfrac{1}{2}I_\mu\otimes I_\nu\otimes[I_\mu,I_\nu])$$
$$=(u1\otimes I_\mu\otimes I_\mu+\tfrac{1}{2}I_\mu\otimes I_\nu\otimes[I_\nu,I_\mu])R_{12}(u).$$

Using

$$[C_2,1\otimes I_\lambda]\equiv[(1\otimes I_\nu+I_\nu\otimes 1)^2,1\otimes I_\lambda]=2c_{\lambda\mu\nu}I_\nu\otimes I_\mu$$

(where C_2 is the quadratic Casimir operator, here evaluated on the tensor product) we obtain

$$R(u)(u1\otimes I_\lambda-\tfrac{1}{4}[C_2,1\otimes I_\lambda])=(u1\otimes I_\lambda+\tfrac{1}{4}[C_2,1\otimes I_\lambda])R(u). \tag{2.8}$$

This equation is now valued on $V_1\otimes V_2$ only, and is the final form of the term of order 1. We shall now use (2.8) to find R_{ab}.

In order to obtain a relation between the $\tau_c(u)$, we substitute the form (2.6) for $R_{ab}(u)$ back into (2.8). Acting on the left with P_d and on the right with P_c, we obtain

$$\tau_d(u)P_d(u+\tfrac{1}{4}(C_2(c)-C_2(d)))(1\otimes I_\lambda)P_c$$
$$=\tau_c(u)P_d(u-\tfrac{1}{4}(C_2(c)-C_2(d))))(1\otimes I_\lambda)P_c. \tag{2.9}$$

But $1\otimes I_\lambda$ is an irreducible tensor operator in the adjoint representation, and so we can apply the Wigner–Eckhart theorem to obtain the general form for group-invariant rational R-matrices acting in irreducible representations a, b of the algebra (where $a\otimes b$ has no multiplicities)

$$R_{ab}(u)=\sum_{c\subset a\otimes b}\tau_c(u)P_c \tag{2.10}$$

where

$$\frac{\tau_c(u)}{\tau_d(u)}=\frac{u+\frac{1}{4}\{C_2(c)-C_2(d)\}}{u-\frac{1}{4}\{C_2(c)-C_2(d)\}} \tag{2.11}$$

for c, d such that $d \subset \text{adjoint} \otimes c$ and $\langle d \| 1 \otimes I_\lambda \| c \rangle$ (the reduced matrix element) not equal to zero.

To deal with (2.11) we first need to know for which c, d (such that $d \subset \text{adjoint} \otimes c$) the reduced matrix element $\langle d \| 1 \otimes I_\lambda \| c \rangle$ vanishes. When we are examining $R_{aa}(u)$ (*that is to say, R acting in two identical representations $a = b$*) *we can split the* components of $a \otimes a$ into those appearing symmetrically and those appearing antisymmetrically in the tensor product. Now $\langle d \| 1 \otimes I_\lambda + I_\lambda \otimes 1 \| c \rangle$ vanishes, and so

$$\langle d \| 1 \otimes I_\lambda \| c \rangle = \tfrac{1}{2} \langle d \| 1 \otimes I_\lambda - I_\lambda \otimes 1 \| c \rangle.$$

The operator thus has negative parity, and so, for the reduced matrix element to be non-zero, c and d must have opposite parity. We now proceed on the assumption that, conversely, when c and d have opposite parity, the matrix element is non-zero. This is certainly true when $a \otimes a$ only contains two states of weight ω_c, since the highest weight ω_c of c is chosen to be orthogonal to the state of the same weight in d:

$$\langle \omega_c | 1 \otimes I_\mu + I_\mu \otimes 1 | \omega_d \rangle = 0.$$

Our system of equations (2.11) then applies to all c, d of opposite parity such that $d \subset \text{adjoint} \otimes c$.

For an R-matrix to exist, it is necessary that this system of equations have a solution. In general, however, the system will be overdetermined. We now proceed to investigate the existence and uniqueness of solutions of this system.

Existence. We check this by forming the representations $c \subset a \otimes a$ into a network. Starting with the representation of highest weight $\Omega = 2\omega_a$, where ω_a is the highest weight of a, we write $c \nrightarrow d$ whenever $d \subset c \otimes \text{adjoint}$ and c, d have opposite parity, and label each such arrow with the number $C_2(c) - C_2(d)$. The set (2.11) is consistent, and thus R is well defined, if and only if, for every pair of representations $p, q \subset a \otimes a$, the set of numbers attached to the arrows on each possible route from p to q is the same. This is the same as saying that all closed paths on the network must give $\tau_p / \tau_p = 1$ for consistency.

Uniqueness. The network described is always connected, since the highest weights of the components of $a \otimes a$ differ by positive roots, and are linked by $1 \otimes I_\mu - I_\mu \otimes 1$. *Thus (if R exists) any one τ_d is sufficient to determine all of the others. Hence R is* defined up to an overall factor, dependent on u. We will choose this factor so that the coefficient of the representation with highest weight Ω is one. Note also that, as a result of (2.11), $\lim_{u \to \infty} R(u) = 1$. In addition, we are free to rescale u: we see that $R(Ku)$ is also a solution of (1.1) for any constant K.

We have not been able to formulate a general method for determining whether or not a given network is consistent. However, we have been able to calculate a large number of specific examples. Consistent networks which reproduce known R-matrices include those for vector representations of SO(N) [8] and Sp($2n$) [9], for spinor representations of SO(N) [10], for various other representations of SO(N) and Sp($2n$) [11–13], and for the defining representations of all of the exceptional groups [6] except E_8. Also consistent are the networks for those representations of A_n for which unitary R-matrices have been calculated [3] (the completely symmetric and antisymmetric representations).

At this stage we mention the essential difference between our results and those of Kulish *et al* for the A_n series. The point is that their equation is obtained without requiring R to be unitary. This is done by setting V_3 to be the vector representation ($\square$ in the usual Young tableaux notation); $R_{a\square}$ is known for any representation a of A_n, and is linear in $1/u$, so that X vanishes. Their equation for R is exact for all v; ours is only true in the limit $v \to \infty$. However, because X vanishes, they also have to take into account terms symmetric in $\mu \leftrightarrow \nu$ in (2.7), which involve the symmetric third order Casimir operator of A_n, $d_{\lambda\mu\nu}$. This adds terms

$$\frac{\langle d \| d_{\mu\nu\lambda} I_\nu \otimes I_\mu \| c \rangle}{\langle d \| 1 \otimes I_\lambda \| c \rangle} \tag{2.12}$$

to both numerator and denominator of (2.11), and any representations for which these do not always vanish cannot have unitary R-matrices, since unitarity requires $\tau_c(u)\tau_c(-u) = 1$. The effect is that for those representations of A_n for which unitary R-matrices exist, our equation is the same as theirs, and can be used to obtain those unitary solutions given in their paper. When (2.12) is non-zero for some c, d, and a unitary solution does not exist, our method has nothing to say about the solution.

As a brief example we give here the network corresponding to R_{aa} for a the defining (seven dimensional) representation of G_2, which is one of the R for exceptional groups previously found using the (analytic) Bethe ansatz [6]. Labelling representations in terms of a basis of fundamental weights so that $a = (1, 0)$ the network is then

$$\begin{array}{l} (2,0)_S \to^1 (0,1)_A \to^6 (0,0)_S \\ \;\;\downarrow_4 \\ (1,0)_A. \end{array}$$

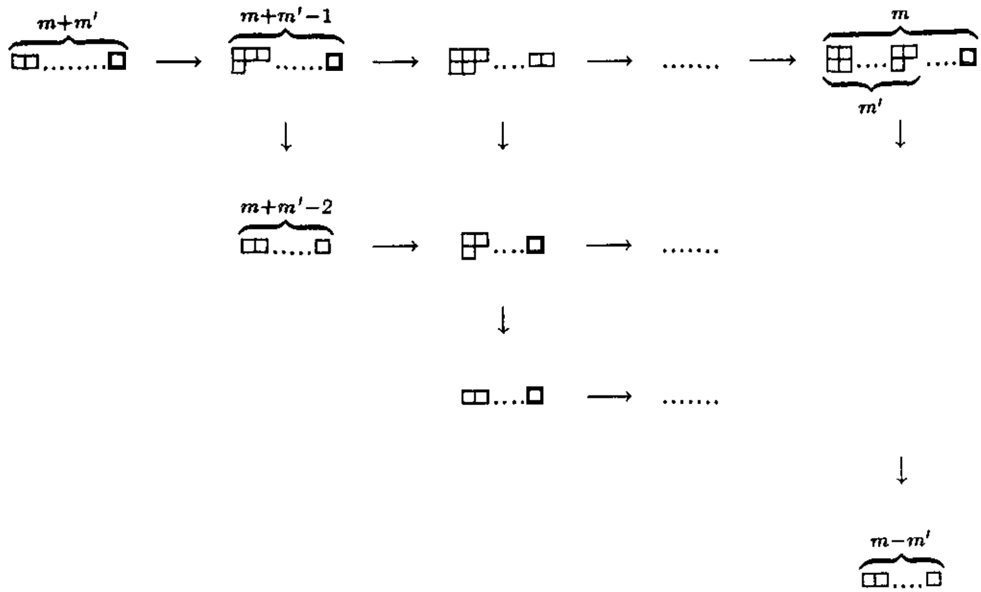
This also illustrates the obvious fact that any network which is a tree is consistent.

We have also found some new R-matrices using this method, including those for some representations of A_n whose Young tableaux are rectangular (although we have not shown that the networks of all such representations are consistent) and for completely symmetric representations of SO(N), which we give shortly as an example. We have tested many other networks (with the help of the LiE computer algebra package [14]) and found them mostly to be inconsistent. All of our results for individual cases agree with the general characterization found by Drinfeld, which will be explained in section 3.

Now consider the cases where $a \neq b$. We have no general method for determining when $\langle d \| 1 \otimes I_\lambda \| c \rangle \neq 0$. However, an intriguing fact to emerge from the study of the $a = b$ networks is that, in all of the consistent examples, whenever $c \subset \text{adjoint} \otimes d$, c and d have opposite parity: in other words, it seems that for consistent networks the parity principle is redundant. We would like to emphasize that this is not true of inconsistent networks, and that it remains only a conjecture for consistent networks. If we go ahead and analyse the consistency of $a \neq b$ networks on the assumption that, if the network is going to be consistent, $\langle d \| 1 \otimes I_\lambda \| c \rangle \neq 0$ whenever $c \subset \text{adjoint} \otimes d$, we obtain matrices R_{ab}, all of which agree both with Drinfeld's classification and, where these exist, with R-matrices found in the papers listed above.

As an illustrative example of new R-matrices, we now calculate the R-matrices in symmetric, traceless representations of SO(N). These could be used to solve the generalization of the XXX magnet in which the (isotropically coupled) spins take arbitrary directions in N dimensions, as advocated by Reshetikhin [12].

Let m, m' be the representations with highest weight $(m, 0, \ldots, 0)$ and $(m', 0, \ldots, 0)$ (with respect to a basis of fundamental weights), where $m \geq m'$. For these representations, the network described is shown below:



(where representations are denoted by the usual Young tableaux, with a trace removed from all symmetric indices). The differences of the Casimirs have not been shown on this diagram. They can be calculated easily using the inverse Cartan matrix, and are found to satisfy the given requirement, i.e. that the rectangles in the network commute. Substituting their values into (2.11) we find that the R-matrix obtained from (2.10), (2.11) is

$$R_{mm'}(u)=\sum_{k=0}^{m'}\sum_{q=0}^{k}\left(\prod_{r=1}^{k}[m+m'+2-2r]\prod_{s=1}^{q}[m+m'+N-2s-2]\right)P_{(m+m'-2k,k-q,0,\ldots,0)}$$

where we have introduced the notation

$$[t]\equiv\frac{u+(t/4)}{u-(t/4)}.$$

This agrees with the R-matrices for $\{m, m'\}=\{1,2\}$, $\{1,3\}$ and $\{2,2\}$ calculated [13] using the fusion procedure [3].

Many of the R-matrices which have already been found are in the form of factorized S-matrices for the interaction of massive particles in integrable models in $1+1$ dimensions. In addition to satisfying (1.1) and unitariy, the S-matrices must also satisfy crossing symmetry. This requires that

$$S_{aa}(\theta)=(S_{aa}(i\pi-\theta))^{T}$$

where θ is the rapidity, and T means transpose and conjugate in the first space, so that if i, j and k, l label states in the incoming and outgoing representations respectively, and $(v_i)^*=v^i$, then $(S^{kl}_{ij})^T=S^{lj}_{ki}$. Now $S(0)$ acts as the identity $\delta^k_i\delta^l_j$, and so the crossed version of $S(0)$ is $\delta^k_l\delta^i_j\propto P_0$ (the singlet representation). Hence we need $S(i\pi)\propto P_0$. But where P_0 is present in the decomposition we note that the tensor product of the singlet and adjoint representations is just the adjoint representation, and so in our definition (2.10), (2.11) we must use our freedom to rescale u to put

$$\theta = \frac{4\mathrm{i}\pi}{C^{\mathrm{Adj}}}\, u$$

where $C_{\mathrm{Adj}} = C_2(\text{adjoint})$.

3. The Yangian construction

For full details of the Yangian, we refer the reader to Drinfeld [4]. However, the salient points are these. The Yangian $Y(\mathscr{A})$ of a Lie algebra $\mathscr{A}$ is obtained by setting $h = 1$ in the algebra which is the unique quantization of the co-Poisson Hopf whose classical r-matrix is

$$r(u) = \frac{1}{u}\, I_\mu \otimes I_\mu.$$

It is a Z_2-graded algebra with generators I_λ (grade zero) and J_λ (grade one). The generators I_λ satisfy the usual commutation relations of the Lie algebra $\mathscr{A}$,

$$[I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu \tag{3.1}$$

whilst the J_λ satisfy

$$[I_\lambda, J_\mu] = c_{\lambda\mu\nu} J_\nu \tag{3.2}$$

and

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha, I_\beta, I_\gamma\} \tag{3.3}$$

where

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \tfrac{1}{24} c_{\lambda\alpha i} c_{\mu\beta j} c_{\nu\gamma k} c_{ijk} \qquad \{x_1, x_2, x_3\} = \sum_{i\neq j\neq k} x_i x_j x_k.$$

For $\mathscr{A} = sl(2)$, (3.3) is replaced by a different relation:

$$[[J_\lambda, J_\mu], [I_r, J_s]] + [[J_r, J_s], [I_\lambda, J_\mu]] = (a_{\lambda\mu\nu\alpha\beta\gamma} c_{rs\nu} + a_{rs\nu\alpha\beta\gamma} c_{\lambda\mu\nu})\{I_\alpha, I_\beta, J_\gamma\}.$$

The co-multiplication is given by

$$\Delta(I_\lambda) = I_\lambda \otimes 1 + 1 \otimes I_\lambda \qquad \Delta(J_\lambda) = J_\lambda \otimes 1 + 1 \otimes J_\lambda + \tfrac{1}{2} c_{\lambda\mu\nu} I_\nu \otimes I_\mu. \tag{3.4}$$

Equation (3.3) is determined by requiring that the coproduct defined in (3.4) is a homomorphism consistent with (3.1), (3.2).

It is then possible to define an automorphism $T_u : Y(\mathscr{A}) \to Y(\mathscr{A})$ for any $u \in \mathbb{C}$ by

$$T_u(J_\lambda) = J_\lambda + u I_\lambda \qquad \text{and} \qquad T_u(I_\lambda) = I_\lambda.$$

Further, defining $T_{u,v} : Y(\mathscr{A}) \otimes Y(\mathscr{A}) \to Y(\mathscr{A}) \otimes Y(\mathscr{A})$ by

$$T_{u,v} = T_u \otimes T_v$$

there then exists a formal R-matrix, $R(u)$, satisfying $T_{v,w} R(u) = R(u + w - v)$, which expresses the non-cocommutativity of the Yangian through

$$T_{0,u} \Delta'(x) = R(u)^{-1} (T_{0,u} \Delta(x)) R(u) \quad (x \in Y(\mathscr{A})). \tag{3.5}$$

In this equation, which is valued in $Y(\mathscr{A})\otimes Y(\mathscr{A})$, Δ' is the product of Δ and the transposition operator P. This R satisfies the Yang–Baxter equation, (1.1), and

$$R(u)R(-u)=1.$$

The existence of the automorphism T_u is essential; it is this that allows the existence of $R(u)$. It is clear that a representation of the Yangian gives rise to a representation of $R(u)$ and hence to a matrix solution of the Yang–Baxter equation. So the search for representations of $Y(\mathscr{A})$ is fundamental to the search for solutions of (1.1).

Drinfeld sought [4] to construct representations $\tilde{\rho}$ of the Yangian as follows. Starting from a representation ρ of $\mathscr{A}$,

$$\tilde{\rho}(I_\lambda)=\rho(I_\lambda) \tag{3.6}$$

he then needed to define $\tilde{\rho}(J_\lambda)$ in a way consistent with the defining relations of $Y(\mathscr{A})$. One way of doing this is to set

$$\tilde{\rho}(J_\lambda)=0. \tag{3.7}$$

However, he showed that it is not possible to do this for all irreducible representations. This is because, although $\tilde{\rho}$ is clearly consistent with (3.1), (3.2), it is not, in general, consistent with (3.3). Consistency is only possible for representations in which the right-hand side of (3.3) vanishes. This is the case for the following representations (theorem 7 of Drinfeld [4]), although not necessarily only for these representations [5].

Let n_α be the coefficient of the simple root α in the expansion of the highest root $\alpha_{\max}$, and let $k_\alpha=(\alpha_{\max},\alpha_{\max})/(\alpha,\alpha)$. Let the corresponding fundamental weight be ω_α. The representation of the group with highest weight Ω may then be extended to a representation of the Yangian for (i) $\Omega=\omega_\alpha$ when $n_\alpha=k_\alpha$ and (ii) $\Omega=t\omega_\alpha$ when $n_\alpha=1$ (t a positive integer).

We now note that the results of section 2 can also be derived from the Yangian. We assume $\tilde{\rho}$ in the form (3.6), (3.7), but, instead of investigating the consistency of $\tilde{\rho}$ with the defining relations of $Y(\mathscr{A})$, we consider the implications of $\tilde{\rho}$ for (3.5). Substituting $x=J_\lambda$ and $x=I_\lambda$ respectively in (3.5), we see that $R_{ab}(u)\equiv\rho_a\otimes\rho_b(R(u))$ must satisfy

$$\begin{aligned} R_{ab}(u)(u1\otimes\rho_b(I_\lambda)-\tfrac{1}{2}c_{\lambda\mu\nu}\rho_a(I_\nu)\otimes\rho_b(I_\mu)) \\ =(u1\otimes\rho_b(I_\lambda)+\tfrac{1}{2}c_{\lambda\mu\nu}\rho_a(I_\nu)\otimes\rho_b(I_\mu))R_{ab}(u) \end{aligned} \tag{3.8}$$

and

$$[R_{ab}(u),1\otimes\rho_b(I_\lambda)+\rho_a(I_\lambda)\otimes1]=0 \tag{3.9}$$

where 1 is the appropriate representation of the identity. Equation (3.9) is just (2.5), whilst equation (3.8) is essentially theorem 4 of Drinfeld's paper, and coincides with (2.8). The general A_n case is dealt with in theorem 9 of his paper, and reproduces the results of Kulish *et al.*

The rext of the analysis of section 2 now follows through.

4. Conclusions

We have presented a general formula for the spectral decomposition of unitary R-matrices in irreducible representations, where they exist, together with a necessary condition that the representations must satisfy for this to be so. Unfortunately we have not been able to develop a general method for solving this condition, and so have usually had to proceed case by case. In an alternative approach, without actually deducing the form of R, Drinfeld presented a set of representations for which he proved that R must exist. We believe that the two conditions coincide, so that we can now compute all such R matrices, but we are unable to give a general proof.

This lack of generality also means that other facts to emerge from the study of examples must remain conjectures: for instance, that in consistent networks $c \subset \text{adjoint} \otimes d \Rightarrow c, d$ have opposite parity, and that (following Drinfeld's comments [4]) the network for a the adjoint representation is always inconsistent.

One way to construct R-matrices is the fusion procedure [3] (or see Jimbo's review [2] or the author's paper [13]), which is a method for constructing new R-matrices from existing ones. In all cases we have examined, analysis of how the fusion procedure can be applied to existing R-matrices shows that it produces R-matrices for precisely those irreducible representations specified in sections 2 and 3.

In addition, the fusion procedure can be used to produce R-matrices in reducible representations [13], the decomposition of some of which is known [6, 11]. It would be interesting to try to extend the methods of section 2 to reducible representations, or alternatively to discover whether the representations in which R-matrices can be found using the fusion procedure correspond to all possible representations of $\mathscr{A}$, reducible and irreducible, which can be extended to representations of $Y(\mathscr{A})$. However, the connections between the fusion procedure, the networks of section 2 and the representation theory of the Yangian currently remain to be explored.

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Note added. Since submission of this paper, similar ideas about networks of representations have appeared in [15] in the context of trigonometric R-matrices.

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